

MAS212 Linear Algebra I Exercises 7 (with solutions)

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Assessed submission deadline 5pm on Friday 24th November 2006 in the blue box on the ground floor.

Your submission must have your FULL NAME and STUDENT NUMBER PRINTED CLEARLY at the top and separate sheets must be stapled together.

Marks out of 100 are shown in square brackets after each question number.

Do not use echelon methods for these exercises – that comes later!

1. [20] Determine explicitly the maximum number of linearly independent rows and columns, and hence the row rank and column rank, of each of the following matrices over the real field $\mathbb{R}$:

$$(a) \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, (b) \begin{bmatrix} 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & -1 \\ -1 & 1 & -1 & 2 \end{bmatrix}.$$

Solution

(a) **[10]** The columns are linearly independent since one is not a scalar multiple of the other, therefore the column rank is 2.

Similarly, any two rows are linearly independent, therefore the row rank is at least 2. Check whether all three rows are linearly independent by solving $a(1, 2) + b(3, 4) + c(5, 6) = (0, 0)$ for $a, b, c \in \mathbb{R}$, i.e. solve

$$\begin{aligned} a + 3b + 5c &= 0 \\ 2a + 4b + 6c &= 0 \end{aligned}$$

One solution is obviously $a = b = c = 0$, but two equations for three variables cannot have a unique solution, so there are nonzero solutions. (To be specific, the general solution can be written as $a = c$, $b = -2c$, so a nonzero solution is $a = c = 1$, $b = -2$.) Therefore the rows are not all linearly independent and the row rank is also 2 (as it must be).

(b) **[10]** By inspection, $\text{row}(1) - \text{row}(2) = \text{row}(3)$, but any two rows are linearly independent since none is a scalar multiple of another, therefore the row rank is 2.

Also by inspection, $\text{col}(1) = \text{col}(2) - \text{col}(4)$ so the column rank is less than 4. Check whether the first three columns are linearly independent by solving

$$a \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + b \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which implies

$$\begin{aligned} 2a + 3b &= 0 \\ 3a + 2b + c &= 0 \\ -a + b - c &= 0. \end{aligned}$$

Adding the last two equations implies that $2a + 3b = 0$, which is identical to the first equation, so the equations are not independent and therefore do not have a unique solution; they are equivalent to two equations for three unknowns. The first equation is satisfied by $a = 3k$, $b = -2k$. Then the last equation implies that $c = -5k$. Hence, a nonzero solution of the equations is

$a = 3, b = -2, c = -5$ and the first three columns are not linearly independent. Hence the column rank is less than 3. Any two columns are clearly linearly independent, therefore the column rank is 2 (as it must be).

► **Maple Solution**

2. [10] Determine the rank of the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ over (a) the rational field $\mathbb{Q}$ and (b) the Boolean field $\mathbb{F}_2$.

▼ **Solution**

(a) [5] Over $\mathbb{Q}$, the rows (and columns) are linearly independent since $a \text{ row}(1) + b \text{ row}(2) + c \text{ row}(3) = 0 \Rightarrow a + b = 0, b + c = 0, a + c = 0 \Rightarrow a = b = c = 0 \Rightarrow$ the rank is maximal, i.e. 3. Alternatively, use columns.

(b) [5] Over $\mathbb{F}_2$, $\text{row}(3) = \text{row}(1) + \text{row}(2)$ by inspection, but any two rows are linearly independent since none is a scalar multiple of another, therefore the rank is 2. [Note that $1 + 1 = 0$ in $\mathbb{F}_2$.] Alternatively, use columns.

► **Maple Solution**

3. [15] Use rank computations to determine the nature of the solution space of the matrix equation $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$ over $\mathbb{R}$, i.e. whether a solution exists and if so what the dimension of the solution space is. Verify this by solving (if possible) the corresponding system of simultaneous linear equations.

▼ **Solution**

The rank of the square matrix is 2 because $\text{row}(1) + \text{row}(3) = 2 \text{ row}(2)$ so the rank is not maximal but any two rows are linearly independent.

However, this is no longer true when the column vector is adjoined, so check that the three rows of the augmented matrix are linearly independent by solving $a(1, 2, 3, 2) + b(3, 4, 5, 3) + c(5, 6, 7, 5) = (0, 0, 0, 0)$, i.e.

$$\begin{aligned} a + 3b + 5c &= 0, \\ 2a + 4b + 6c &= 0, \\ 3a + 5b + 7c &= 0, \\ 2a + 3b + 5c &= 0. \end{aligned}$$

Eliminating a from the first and third equations implies $4b + 8c = 0$, i.e. $b + 2c = 0$ and eliminating a from the second and fourth equations implies $b + c = 0$. These two equations together have the unique solution $b = c = 0$. Then any of the original equations implies that also $a = 0$. Hence $a = b = c = 0$ is the unique solution and the rows are linearly independent, so the augmented matrix has rank 3.

Since the ranks of the augmented and original matrices differ, there is *no solution*, i.e. the equations are inconsistent.

The scalar equations are

$$\begin{aligned}x + 2y + 3z &= 2, \\3x + 4y + 5z &= 3, \\5x + 6y + 7z &= 5.\end{aligned}$$

Eliminating x between the first and second equations gives $2y + 4z = 3$ and eliminating x between the first and third equations gives $4y + 8z = 5$. These equations are inconsistent and so have no solution; for example, multiplying the first new equation by 2 and subtracting the second in an attempt to eliminate either y or z leads to the false assertion that $0 = 1$.

► Maple Solution

4. [25] Compute the determinant of each of the following matrices and if its inverse exists then compute it *and check that it satisfies the definition of the inverse of a matrix*:

$$(a) \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad (b) \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix}, \quad (c) \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ 0 & 3 & 1 \end{bmatrix}.$$

▼ Solution

(a) [10] By diagonal expansion, $\det(A) = (0 + 1 + 0) - (0 + 0 + 0) = 1$. Since this is nonzero, the inverse exists and is the adjoint divided by the determinant. First, the matrix of minors is

$$\begin{bmatrix} 0-0 & 0-1 & 0-0 \\ 0-0 & 0-0 & 0-1 \\ 1-0 & 0-0 & 0-1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix}.$$

From this, changing signs as appropriate and transposing gives the adjoint and dividing by the determinant give the inverse as

$$\frac{1}{1} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

Check:

$$\frac{1}{1} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) [5] From the previous question, the rank of this matrix is not maximal so the determinant is zero and it has no inverse.

(c) [10] By diagonal expansion, $\det(A) = (0 + 0 + 18) - (0 + 4 + 9) = 5$. Since this is nonzero, the inverse exists and is the adjoint divided by the determinant. First, the matrix of minors is

$$\begin{bmatrix} 0-9 & 2-0 & 6-0 \\ 2-9 & 1-0 & 3-0 \\ 6-0 & 3-6 & 0-4 \end{bmatrix} = \begin{bmatrix} -9 & 2 & 6 \\ -7 & 1 & 3 \\ 6 & -3 & -4 \end{bmatrix}.$$

From this, changing signs as appropriate and transposing gives the adjoint and dividing by the determinant give the inverse as

$$\frac{1}{5} \begin{bmatrix} -9 & 7 & 6 \\ -2 & 1 & 3 \\ 6 & -3 & -4 \end{bmatrix}.$$

Check:

$$\frac{1}{5} \begin{bmatrix} -9 & 7 & 6 \\ -2 & 1 & 3 \\ 6 & -3 & -4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

► **Maple Solution**

5. [30] (a) For two general 2×2 matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ show explicitly that $|AB| = |A| |B|$.

▼ **Solution [10]**

- (a) Let $A := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B := \begin{bmatrix} e & f \\ g & h \end{bmatrix}$. Then $AB = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$ so $|AB| = (ae+bg)(cf+dh) - (af+bh)(ce+dg) = aedh + bgcf - afdg - bhce$, and also $|A| |B| = (ad-bc)(eh-fg) = aedh + bgcf - afdg - bhce$.

► **Maple Solution**

- (b) Evaluate the determinant $\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 4 & 0 \\ 5 & 5 & 5 & 5 & 5 \end{vmatrix}$ efficiently.

▼ **Solution [10]**

The value of a triangular determinant is the product of its diagonal elements, so this determinant is $1 \times 2 \times 3 \times 4 \times 5 = 5! = 120$.

► **Maple Solution**

- (c) An $n \times n$ integer matrix A of the form $A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix}$, in which the entries of

successive rows are shifted to the right one position (modulo n), is called a *circulant matrix* and its determinant $|A|$ is called a *circulant* or a *cyclic determinant*. Prove that $|A|$ contains $\sum_{i=0}^{n-1} a_i$ as a factor. [Hint: Consider the sum of all the rows or all the columns. Use the properties of determinants to rewrite $|A|$ so that every element in one of the rows or columns is the same and then

factor it out. Remark: In fact, the elements of A could be elements of any unique factorization domain, of which the integers provides the simplest example, but they cannot be field elements.]

▼ **Solution [10]**

Add all columns but the first to the first column. This does not change the value of the

determinant but every element in the first column becomes equal to $\sum_{i=0}^{n-1} a_i$. A determinant is

linear in each of its rows and columns, so a common factor of every element of a column can be factored out as a scalar multiple.

Optional Extras (no marks)

6. (a) If Λ is an $n \times n$ diagonal matrix of the form $\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$ with $\lambda_i \neq 0, 1 \leq i \leq n$,

prove that $\Lambda^r = \begin{bmatrix} \lambda_1^r & 0 & \cdots & 0 \\ 0 & \lambda_2^r & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^r \end{bmatrix}$ for any integer r (and in particular for $r = -1$).

▼ **Solution**

Suppose the result is true for some integer r . Then

$$\Lambda^{r+1} = \Lambda^r \Lambda = \begin{bmatrix} \lambda_1^r & 0 & \cdots & 0 \\ 0 & \lambda_2^r & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^r \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1^{r+1} & 0 & \cdots & 0 \\ 0 & \lambda_2^{r+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^{r+1} \end{bmatrix} \text{ and the result is}$$

true for $r + 1$. The result is true by definition for $r = 1$, therefore by induction it is true for all integer $r \geq 1$. The result is true for $r = 0$ provided we interpret Λ^0 as the identity matrix for any square matrix Λ and the result is true for $r = -1$ because

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\lambda_n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Finally, suppose the result is true for some (negative) integer r . Then

$$\Lambda^{r-1} = \Lambda^r \Lambda^{-1} = \begin{bmatrix} \lambda_1^r & 0 & \cdots & 0 \\ 0 & \lambda_2^r & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^r \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\lambda_n} \end{bmatrix} = \begin{bmatrix} \lambda_1^{r-1} & 0 & \cdots & 0 \\ 0 & \lambda_2^{r-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^{r-1} \end{bmatrix} \text{ and the}$$

result is true for $r-1$. Therefore by induction the result is true for all negative integer r and hence all integer r .

(b) Let Ω be an $n \times n$ diagonal matrix of the form $\Omega = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega & 0 & \cdots & 0 \\ 0 & 0 & \omega^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \omega^{n-1} \end{bmatrix}$ where ω is an n^{th} root

of unity, i.e. one of the n distinct complex solutions of the equation $x^n = 1$. If A is the circulant matrix defined in Question 5(c) above, verify that $\Omega^{-1} A \Omega$ is a circulant matrix with elements $a_i \omega^i$ and that

$|\Omega^{-1} A \Omega| = |A|$. Hence, show that $\sum_{i=0}^{n-1} a_i \omega^i$ is also a factor of $|A|$ and that the circulant $|A|$ is

precisely equal to the product of the n such factors corresponding to the n distinct n^{th} roots of unity ω_j , $1 \leq j \leq n$. Verify this explicitly for the circulants with $n = 2$ and $n = 3$.

Solution

Post-multiplying A by Ω multiplies the j^{th} column by ω^{j-1} to give

$$\begin{bmatrix} a_0 & a_1 \omega & a_2 \omega^2 & \cdots & a_{n-1} \omega^{n-1} \\ a_{n-1} & a_0 \omega & a_1 \omega^2 & \cdots & a_{n-2} \omega^{n-1} \\ a_{n-2} & a_{n-1} \omega & a_0 \omega^2 & \cdots & a_{n-3} \omega^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 \omega & a_3 \omega^2 & \cdots & a_0 \omega^{n-1} \end{bmatrix}. \text{ Using the previous result on powers of a diagonal}$$

matrix shows that pre-multiplying A by Ω^{-1} multiplies the i^{th} row by ω^{1-i} to give

$$\begin{bmatrix} a_0 & a_1 \omega & a_2 \omega^2 & \cdots & a_{n-1} \omega^{n-1} \\ a_{n-1} \omega^{-1} & a_0 & a_1 \omega & \cdots & a_{n-2} \omega^{n-2} \\ a_{n-2} \omega^{-2} & a_{n-1} \omega^{-1} & a_0 & \cdots & a_{n-3} \omega^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 \omega^{1-n} & a_2 \omega^{2-n} & a_3 \omega^{3-n} & \cdots & a_0 \end{bmatrix}. \text{ Finally, using the fact that } \omega^n = 1 \text{ to remove}$$

negative powers (or rationalize the denominators) in the lower triangular part of the matrix gives

$$\begin{bmatrix} a_0 & a_1 \omega & a_2 \omega^2 & \cdots & a_{n-1} \omega^{n-1} \\ a_{n-1} \omega^{n-1} & a_0 & a_1 \omega & \cdots & a_{n-2} \omega^{n-2} \\ a_{n-2} \omega^{n-2} & a_{n-1} \omega^{n-1} & a_0 & \cdots & a_{n-3} \omega^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 \omega & a_2 \omega^2 & a_3 \omega^3 & \cdots & a_0 \end{bmatrix}. \text{ Hence } \Omega^{-1} A \Omega \text{ is a circulant matrix with}$$

elements $a_i \omega^i$. By the general product rule for determinants,

$$|\Omega^{-1} A \Omega| = |\Omega^{-1}| |A| |\Omega| = |\Omega \Omega^{-1}| |A| = |A|. \text{ Then the same addition technique as before}$$

shows that $\sum_{i=0}^{n-1} a_i \omega^i$ is also a factor of $|A|$. There are n such factors corresponding to the n

distinct n^{th} roots of unity ω_j , $1 \leq j \leq n$, and each such factor involves a_0 with coefficient 1, so their product involves a_0^n with coefficient 1. The general properties of determinants, or equivalently consideration of the evaluation of the determinant by recursive Laplace expansion, shows that the circulant also involves a_0^n with coefficient 1. Therefore, the circulant $|A|$ is precisely equal to the product of the n factors corresponding to the n distinct n^{th} roots of unity ω_j , $1 \leq j \leq n$.

Verifying this explicitly for the circulant with $n = 2$ gives $\begin{vmatrix} a & b \\ b & a \end{vmatrix} = a^2 - b^2$ and also

$$(a + b \omega) \Big|_{\omega=1} (a + b \omega) \Big|_{\omega=-1} = a^2 - b^2; \text{ the circulant with } n = 3 \text{ gives}$$

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = a^3 - 3 a b c + c^3 + b^3 \text{ and also}$$

$$(a + b \omega + c \omega^2) \Big|_{\omega=1} (a + b \omega + c \omega^2) \Big|_{\omega=e^{\frac{2}{3} i \pi}} (a + b \omega + c \omega^2) \Big|_{\omega=e^{-\frac{2}{3} i \pi}} = a^3 - 3 a b c + c^3 + b^3.$$

7. (a) Let $\alpha : U \rightarrow V$ and $\beta : V \rightarrow W$ be linear maps. Prove that (i) $\text{rank}(\beta \alpha) \leq \text{rank}(\alpha)$; (ii) $\text{rank}(\beta \alpha) = \text{rank}(\alpha)$ if β is one-to-one; (iii) $\text{rank}(\beta \alpha) = \text{rank}(\beta)$ if α is onto.

Solution

- (i) By definition, $\text{rank}(\alpha) = \dim(\alpha(U))$ and $\text{rank}(\beta \alpha) = \dim(\beta \alpha(U))$. When β is applied after α it can act only on $\alpha(U)$, the image of α , so we can restrict the map β to the domain $\alpha(U) \subseteq V$ and consider $\beta : \alpha(U) \rightarrow W$. Then $\text{im}(\beta) = \beta \alpha(U)$ and $\dim(\beta \alpha(U)) = \dim(\text{im}(\beta)) \leq \dim(\text{im}(\beta)) + \dim(\ker(\beta)) = \dim(\alpha(U))$ by the Dimension Theorem for the map β . Hence $\text{rank}(\beta \alpha) \leq \text{rank}(\alpha)$.
- (ii) If β is one-to-one then $\dim(\ker(\beta)) = 0$, so $\text{rank}(\beta \alpha) = \text{rank}(\alpha)$.
- (iii) If α is onto then $\alpha(U) = V$ and $\beta \alpha(U) = \beta(V) = \text{im}(\beta)$, so $\text{rank}(\beta \alpha) = \text{rank}(\beta)$.

- (b) Let A be an $n \times m$ matrix, P an invertible $m \times m$ matrix and Q an invertible $n \times n$ matrix, all over the same field. Prove that (i) $\text{rank}(AP) = \text{rank}(A)$, (ii) $\text{rank}(QA) = \text{rank}(A)$, (iii) $\text{rank}(QAP) = \text{rank}(A)$, (iv) an invertible matrix has maximal rank.

Solution

Regard the matrices A, P, Q as linear maps. Since P and Q are invertible they are onto and one-to-one.

- (i) Applying result (iii) above with $\beta = A$ and $\alpha = P$ shows that $\text{rank}(AP) = \text{rank}(A)$.
- (ii) Applying result (ii) above with $\beta = Q$ and $\alpha = A$ shows that $\text{rank}(QA) = \text{rank}(A)$.
- (iii) $\text{rank}(QAP) = \text{rank}(QA) = \text{rank}(A)$.
- (iv) Take A to be the identity matrix (which necessarily has maximal rank) in either of (i) or (ii).